Transverse plasma waves and their instability. Part 2

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A dispersion relation is set up for transverse waves in a plasma with an ellipsoidal velocity distribution. It is shown that the most unstable wave has its wave-normal parallel to the shortest axis of the ellipsoid and its vector potential parallel or anti-parallel to the longest axis. The maximum possible amplification rate is then calculated. Ellipsoidal velocity distributions arise in any flow with an anisotropic pressure tensor. If the resulting instability leads to a microscopic redistribution of particle velocities, then the effective transport coefficients of the plasma are changed. In particular it is shown that the effective viscosity is decreased, and becomes dependent on the local gradient of the macroscopic velocity.

1. Introduction

It was shown previously (Kahn 1962, here referred to as I) that plasmas, with a wide variety of anisotropic velocity distributions, can often be unstable to transverse waves. The calculations of I and of the present paper apply to uniform plasmas which are not subject to any externally imposed electric or magnetic fields.

One is likely to meet anisotropic velocity distributions in a variety of fluid dynamical situations, for example in a shear flow, or in a one- or two-dimensional compressive or expansive flow. The anisotropy is such that the velocity distribution of the charged particles becomes locally ellipsoidal, provided only the effective mean free path remains small enough. In an ordinary gas the anisotropy would tend to be removed by inter-particle collisions, and one can use this knowledge to compute the appropriate coefficient of viscosity, and so on. But it may well happen that these collisions are not effective enough. Transverse instabilities then become excited, and they will now determine how rapidly the anisotropy in the velocity distribution will be removed. Arguing thus one can revise the value of the coefficient of viscosity to allow for collective plasma effects. The revised coefficient depends on the local velocity gradient.

Here we shall consider transverse waves in plasmas with an ellipsoidal velocity distribution. At first the direction of their wave vector is allowed to be arbitrary. Next it will be shown that both the unstable wave with the greatest wave-number, and the one with the largest amplification rate, usually have their wave-normal parallel to the shortest axis of the velocity ellipsoid. Finally, these results are used to estimate revised values of the coefficient of viscosity for a plasma in a shear or a compressive flow.

2. The dispersion relation for an arbitrary wave vector

In this calculation one can either take co-ordinate axes parallel to the principal axes of the velocity ellipsoid and have the wave vector direction arbitrary, or else let the velocity ellipsoid have an arbitrary orientation and let the wave vector be parallel to Ox. We choose the latter alternative, and adopt the distribution function

$$\phi(u,v,w) = \Delta^{\frac{1}{2}}\Phi(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv)\dagger \tag{1}$$

for the undisturbed plasma, where

$$\Delta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

The symbols Φ and ϕ are used for the distribution function to save confusion with the element F and its co-factor f in the determinant Δ. The normalization condition on ϕ is that

$$\begin{aligned} 1 &= \iiint_{-\infty}^{\infty} \phi(u,v,w)\,du\,dv\,dw \\ &= \Delta^{\frac{1}{2}} \iiint_{-\infty}^{\infty} \Phi\,du\,dv\,dw. \end{aligned} \tag{2}$$

Let this relation be transformed to the principal axes $Ou'v'w'$ of the ellipsoid, whose equation then becomes

$$\frac{u'^2}{\alpha^2}+\frac{v'^2}{\beta^2}+\frac{w'^2}{\gamma^2} = \text{const.} = Q^2 \text{ say,} \tag{3}$$

where the axes are labelled so that $\alpha^2 \leqslant \beta^2 \leqslant \gamma^2$. Then $\Delta^{\frac{1}{2}} = (\alpha\beta\gamma)^{-1}$, since Δ is invariant with respect to orientation. The normalization condition (2) now takes the form

$$\begin{aligned} 1 &= \frac{1}{\alpha\beta\gamma}\iiint_{-\infty}^{\infty} \Phi\left(\frac{u'^2}{\alpha^2}+\frac{v'^2}{\beta^2}+\frac{w'^2}{\gamma^2}\right) du'\,dv'\,dw' \\ &\equiv 4\pi\int_0^{\infty} \Phi(Q^2)\,Q^2 dQ. \end{aligned} \tag{4}$$

Let us also define

$$\left.\begin{aligned} \nu_1 &= 4\pi\int_0^{\infty} \Phi(Q^2)\,Q\,dQ, \\ \nu_0 &= 4\pi\int_0^{\infty} \Phi(Q^2)\,dQ. \end{aligned}\right\} \tag{5}$$

We recall from paper I, equation (8), the definition

$$I_{rs\ldots}(q) = \int_{-\infty}^{\infty} \frac{du}{u-q}\frac{\partial}{\partial u}\iint_{-\infty}^{\infty} rs\ldots\,\phi(u,v,w)\,dv\,dw,$$

with some slight changes in notation, and we shall work out expressions for I_1, I_v, I_w, I_{vv}, I_{vw} and I_{ww}, which occur in the dispersion relation, all evaluated at

† From now on, if the function Φ has the argument $Au^2+\ldots+2Fvw+\ldots$, then we shall simply write Φ. When the argument is different it will be given explicitly. The same applies to the functions Φ', Ψ and Ψ'.

$q = 0$. It was shown in I that for a centrally symmetrical velocity distribution I_v and I_w are pure imaginary on the imaginary axis, and therefore at $q = 0$, while the other I-functions mentioned are real there. Until further notice the phase velocities of all the I-functions considered here will be $q = 0$. Then we have that

$$I_1 = \Delta^{\frac{1}{2}} \iiint_{-\infty}^{\infty} \frac{2}{u} (Au + Hv + Gw)\, \Phi'\, du\, dv\, dw. \tag{6}$$

But
$$0 = \int_{-\infty}^{\infty} \frac{du}{u} \iint_{-\infty}^{\infty} \frac{\partial}{\partial v} \Phi\, dv\, dw$$
$$= \iiint_{-\infty}^{\infty} \frac{2}{u} (Hu + Bv + Fw)\, \Phi'\, du\, dv\, dw, \tag{7}$$

and similarly
$$0 = \iiint_{-\infty}^{\infty} \frac{2}{u} (Gu + Fv + Cw)\, \Phi'\, du\, dv\, dw. \tag{8}$$

With the help of (7) and (8), and after a little algebra, (6) becomes

$$I_1 = \frac{2\Delta^{\frac{3}{2}}}{a} \iiint_{-\infty}^{\infty} \Phi'\, du\, dv\, dw, \tag{9}$$

where a is the co-factor of A in the determinant Δ. (A similar definition will be adopted for the symbols b, c, f, g and h to be used later.) Referred to the principal axes of the velocity ellipsoid, and then expressed in terms of Q, the integral in (9) becomes

$$4\pi\alpha\beta\gamma \int_0^{\infty} \Phi'(Q^2)\, Q^2 dQ = -2\pi\alpha\beta\gamma \int_0^{\infty} \Phi(Q^2)\, dQ = -\nu_0/2\Delta^{\frac{1}{2}},$$

and (9) then gives
$$I_1 = -\nu_0 \Delta/a. \tag{10}$$

Next I_v, being pure imaginary, is given by

$$I_v = \pi i \left[\frac{\partial}{\partial u} \iint_{-\infty}^{\infty} v\phi(u, v, w)\, dv\, dw \right]_{u=0}$$
$$= 2\pi i \Delta^{\frac{1}{2}} \iint_{-\infty}^{\infty} v(Hv + Gw)\, \Phi'(Bv^2 + 2Fvw + Cw^2)\, dv\, dw. \tag{11}$$

To calculate I_v we notice that after integrating by parts with respect to v, and expressing the resultant integrand in terms of Q,

$$2\Delta^{\frac{1}{2}} \iint_{-\infty}^{\infty} v(Bv + Fw)\, \Phi'(Bv^2 + 2Fvw + Cw^2)\, dv\, dw$$
$$= -\Delta^{\frac{1}{2}} \iint_{-\infty}^{\infty} \Phi(Bv^2 + 2Fvw + Cw^2)\, dv\, dw$$
$$= -\frac{2\pi\Delta^{\frac{1}{2}}}{a^{\frac{1}{2}}} \int_0^{\infty} \Phi(Q^2)\, Q\, dQ = -\frac{\Delta^{\frac{1}{2}}\nu_1}{2a^{\frac{1}{2}}}, \tag{12}$$

while
$$2\Delta^{\frac{1}{2}} \iint_{-\infty}^{\infty} v(Fv + Cw)\, \Phi'(Bv^2 + 2Fvw + Cw^2)\, dv\, dw$$
$$= \Delta^{\frac{1}{2}} \iint_{-\infty}^{\infty} v \frac{\partial}{\partial w} \Phi(Bv^2 + 2Fvw + Cw^2)\, dv\, dw = 0. \tag{13}$$

With the help of (12) and (13), and again after some algebra, one obtains

$$I_v = (i\pi h/2a^{\frac{3}{2}})\,\Delta^{\frac{1}{2}}\nu_1, \tag{14}$$

and similarly,

$$I_w = (i\pi g/2a^{\frac{3}{2}})\,\Delta^{\frac{1}{2}}\nu_1. \tag{15}$$

It remains to calculate I_{vv}, I_{vw} and I_{ww}. This has to be done by a slightly round-about method. We note that

$$\begin{aligned} HI_{uu}+BI_{uv}+FI_{uw} &= \Delta^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial}{\partial u}\left\{u\iint_{-\infty}^{\infty}(Hu+Bv+Fw)\,\Phi\right\}dv\,dw \\ &= \tfrac{1}{2}\Delta^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial}{\partial u}\left\{u\iint_{-\infty}^{\infty}\frac{\partial}{\partial v}\Psi\right\}dv\,dw = 0 \end{aligned} \tag{16}$$

where we define $\Psi' \equiv \Phi$, provided only that the function Ψ tends to a finite limit as its argument tends to $+\infty$. Similarly

$$GI_{uu}+FI_{uv}+CI_{uw} = 0. \tag{17}$$

But

$$\begin{aligned} I_{uu} &= \iiint_{-\infty}^{\infty}\frac{1}{u}\frac{\partial}{\partial u}(u^2\phi)\,du\,dv\,dw \\ &= \iiint_{-\infty}^{\infty}\phi\,du\,dv\,dw = 1, \end{aligned} \tag{18}$$

and it now follows from (16) and (17) that

$$I_{uv} = h/a \quad \text{and} \quad I_{uw} = g/a.$$

To find I_{vv}, I_{vw} and I_{ww} observe that

$$\begin{aligned} HI_{uv}+BI_{vv}+FI_{vw} &= \Delta^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial}{\partial u}\iint_{-\infty}^{\infty}v(Hu+Bv+Fw)\,\Phi\,dv\,dw \\ &= -\tfrac{1}{2}\Delta^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{du}{u}\frac{\partial}{\partial u}\iint_{-\infty}^{\infty}\Psi\,dv\,dw = -\frac{\Delta^{\frac{3}{2}}}{a}\iiint_{-\infty}^{\infty}\Phi\,du\,dv\,dw, \end{aligned} \tag{19}$$

by analogy with (6) and (9). The normalization condition, applied to (19), yields

$$HI_{uv}+BI_{vv}+FI_{vw} = -\Delta/a. \tag{20}$$

Further, by the usual method of integration by parts, it can be shown that

$$GI_{uv}+FI_{vv}+CI_{vw} = 0. \tag{21}$$

After inserting the value for I_{uv} from equation (18), (20) and (21) can be solved to give

$$I_{vv} = -C\Delta/a^2+h^2/a^2 \quad \text{and} \quad I_{vw} = F\Delta/a^2+hg/a^2, \tag{22, 23}$$

and by analogy with (22)

$$I_{ww} = -B\Delta/a^2+g^2/a^2. \tag{24}$$

According to I, equation (27), the dispersion relation reads

$$\lambda^2+\lambda(\mathscr{I}_{vv}+\mathscr{I}_{ww})+\mathscr{I}_{vv}\mathscr{I}_{ww}-\mathscr{I}_{vw}^2 = 0 \tag{25}$$

where $\lambda \equiv k^2/k_0^2+1$, $k_0^2 \equiv 4\pi Ne^2/mc^2$, $\mathscr{I}_{rs} \equiv I_{rs}-I_r I_s/I_1$, and where r and s can stand for v or w. With the relations (10), (14), (15), (22), (23), and (24) we now find after some simplification that

$$\mathscr{I}_{vv}+\mathscr{I}_{ww} = -\frac{1}{a^2}\left\{(B+C)\Delta+\left(\frac{\pi^2\nu_1^2}{4\nu_0}-1\right)(h^2+g^2)\right\}, \tag{26}$$

and
$$\mathscr{I}_{vv}\mathscr{I}_{ww}-\mathscr{I}_{vw}^2 = \frac{\Delta}{a^3}\left\{\left(2-\frac{\pi^2\nu_1^2}{4\nu_0}\right)\Delta+\left(\frac{\pi^2\nu_1^2}{4\nu_0}-1\right)Aa\right\}. \tag{27}$$

Relations (26) and (27) can be expressed more conveniently, by considering the determinant

$$\delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \tag{28}$$

whose elements are co-factors of the elements in Δ, and which has the value Δ^2. The co-factor of b in δ is

$$ac-g^2 = B\Delta;$$

similarly
$$ab-h^2 = C\Delta.$$

Thus
$$g^2+h^2 = a(b+c)-(B+C)\Delta, \tag{29}$$

and we can re-write

$$\mathscr{I}_{vv}+\mathscr{I}_{ww} = -a^{-2}\{a(b+c)-(2-\rho)(h^2+g^2)\}, \tag{30}$$

where $\rho \equiv \pi^2\nu_1^2/4\nu_0$.

It is convenient to orient the v and w axes so as to make the co-factor f vanish. This can be done since we know, from I, that the dispersion relation is invariant with respect to a rotation of the co-ordinate axes about the direction of the wave vector. Then (27) becomes

$$\begin{aligned}\mathscr{I}_{vv}\mathscr{I}_{ww}-\mathscr{I}_{vw}^2 &= a^{-3}[(2-\rho)\Delta^2+(\rho-1)aA\Delta] \\ &= a^{-3}[(2-\rho)(abc-bg^2-ch^2)+(\rho-1)abc] \\ &= a^{-3}[abc-(2-\rho)(bg^2+ch^2)],\end{aligned} \tag{31}$$

and the dispersion relation (25) may be written

$$\begin{aligned}D(\lambda) \equiv \lambda^2-\lambda a^{-2}[a(b+c)-(2-\rho)(h^2+g^2)]& \\ +a^{-3}[abc-(2-\rho)(bg^2+ch^2)] = 0.&\end{aligned} \tag{32}$$

3. The largest unstable wave-number and the maximum amplification rate

The discussion in this section applies only to velocity distributions for which ρ does not exceed 2. This is somewhat restrictive, in that an absolute upper limit is found, by Schwartz's inequality, to be given by

$$\rho \equiv \pi^2\nu_1^2/4\nu_0 \leqslant \pi^2/4 \doteqdot 2{\cdot}47;$$

but it may be possible to extend the proof to include all values of ρ up to $\frac{1}{4}\pi^2$. For the present we note that an ellipsoidal Gaussian distribution has

$$\Phi(Q^2) = (2\pi)^{-\frac{3}{2}}e^{-\frac{1}{2}Q^2}$$

and here $\nu_1^2/\nu_0 = 2/\pi$, so that $\rho = \frac{1}{2}\pi < 2$. This case is covered in the present treatment.

Let the axes be labelled so that $a \leqslant b \leqslant c$; then we see at once, from (32), that

$$D(c/a) = (2-\rho)(c-b)g^2/a^3 \geqslant 0 \tag{33}$$

and that
$$D'(c/a) = (c-b)/a + (2-\rho)(h^2+g^2)/a^2 \geqslant 0. \tag{34}$$

Thus neither root of the quadratic equation $D(\lambda) = 0$ can exceed c/a. The largest value of c/a occurs when the xyz-axes are the principal axes of the velocity ellipsoid. and then it equals γ^2/α^2. In that case (33) also becomes an equality. Hence the largest λ is

$$\lambda_{\max} \equiv k_{\max}^2/k_0^2 + 1 = \gamma^2/\alpha^2$$

and so
$$k_{\max} = (k_0/\alpha)(\gamma^2-\alpha^2)^{\frac{1}{2}}. \tag{35}$$

This largest value of the wave-number occurs for a disturbance of zero phase velocity; the corresponding wave vector is parallel to the shortest axis of the ellipsoid (the α-axis) and the vector potential of the transverse wave is polarized parallel to the longest axis (the γ-axis).

We go on to find the most unstable wave and the corresponding instability rate for a plasma with a slightly ellipsoidal velocity distribution. The word slightly means, strictly, that the quantities A–B, B–C, F, G and H are all of order ϵA.

Now an ellipsoidal velocity distribution produces real values of k^2 on the imaginary axis of the complex phase-velocity plane. Close to the origin we have, to first order, that

$$k^2 \equiv k_0^2(\lambda-1) \doteqdot k_0^2(\Lambda - 1 + i\Lambda'\eta), \tag{36}$$

where the phase velocity $q \equiv \xi + i\eta$, and Λ and Λ' are the values of λ and $d\lambda/dq$ at $q = 0$. Since λ is real on the imaginary axis, Λ' is pure imaginary. If we split up

$$\lambda = \lambda_r + i\lambda_i,$$
$$\Lambda' = -i(\partial\lambda_r/\partial\eta)|_{q=0} = i(\partial\lambda_i/\partial\xi)|_{q=0}. \tag{37}$$

The amplification rate corresponding to a given λ is $k\eta$, and is largest when

$$k^2\eta^2 = k_0^2[(\Lambda-1)\eta^2 - (\partial\lambda_i/\partial\xi)_0\eta^3] \tag{38}$$

reaches its maximum, or when

$$\eta = \eta_{\max} = \tfrac{2}{3}(\Lambda-1)/(\partial\lambda_i/\partial\xi)_0. \tag{39}$$

We shall see that $(\partial\lambda_i/\partial\xi)_0$ is positive, so that (38) gives a positive value for $\eta_{\max}$ when $\Lambda > 1$. Further $\Lambda - 1$ is of order ϵ when the velocity distribution is only slightly anisotropic. Thus $(\partial\lambda_i/\partial\xi)_0$ is only required correct to the zeroth order, if η is to be found to the first order in ϵ; this means that it is good enough to find $(\partial\lambda_i/\partial\xi)_0$ for the nearest isotropic plasma. If the velocity distribution of the given plasma is $\phi = \Delta^{\frac{1}{2}}\Phi$, then the nearest isotropic plasma has a distribution function

$$\phi_{\text{iso}} = \Delta^{\frac{1}{2}}\Phi\{\Delta^{\frac{1}{3}}(u^2+v^2+w^2)\}$$

for which
$$\lambda = -I_{vv}$$

and
$$(\partial\lambda/\partial\xi)_{q=0} = -\pi i \frac{\partial^2}{\partial u^2}\iint_{-\infty}^{\infty} v^2 \phi_{\mathrm{iso}}\, dv\, dw\big|_{q=0}$$
$$= -2\pi i \Delta^{\frac{5}{6}} \iint_{-\infty}^{\infty} v^2 \Phi' [\Delta^{\frac{1}{3}} (v^2+w^2)]\, dv\, dw$$
$$= \pi i \Delta^{\frac{1}{2}} \iint_{-\infty}^{\infty} \Phi[\Delta^{\frac{1}{3}} (v^2+w^2)]\, dv\, dw$$
$$= 2\pi^2 i \Delta^{\frac{1}{6}} \int_0^{\infty} \Phi(Q^2)\, Q\, dQ = \tfrac{1}{2} i \nu_1 \pi \Delta^{\frac{1}{6}}, \tag{40}$$

with the help of definition (5). Hence
$$\eta_{\max} = \tfrac{2}{3}(\Lambda-1)/(\pi/2)\, \nu_1 \Delta^{\frac{1}{6}},$$
and the corresponding amplification rate is
$$\varpi_{\max} = k_{\max}\eta_{\max} = \frac{4k_0}{3\pi\,.\,3^{\frac{1}{2}}} \frac{(\Lambda-1)^{\frac{3}{2}}}{\nu_1 \Delta^{\frac{1}{6}}}. \tag{41}$$

We have seen that the largest possible $\Lambda = \gamma^2/\alpha^2$ and that $\Delta = (\alpha^2\beta^2\gamma^2)^{-1}$; hence the maximum possible amplification rate for the given plasma becomes
$$\varpi_{\max\max} = \frac{4k_0}{3\pi\,.\,3^{\frac{1}{2}}} \frac{(\alpha\beta\gamma)^{\frac{1}{3}}}{\nu_1} \left(\frac{\gamma^2}{\alpha^2} - 1\right)^{\frac{3}{2}}$$
$$\doteqdot (8\,.\,2^{\frac{1}{2}}/3\pi\,.\,3^{\frac{1}{2}})\, k_0 \sigma \nu_1^{-1} \epsilon^{\frac{3}{2}}, \tag{42}$$
where $\epsilon \equiv (\gamma-\alpha)/\alpha$, and $\sigma = (\alpha\beta\gamma)^{\frac{1}{3}}$. The most unstable wave has the same wave-normal and sense of polarization as the wave with the largest wave-number.

This calculation applies to transverse instabilities in an electron plasma with an immobile background charge. But with minor changes it can be adapted to the case of a plasma which is unstable because the ion distribution is ellipsoidal and in which the electrons have an isotropic velocity distribution. Then the largest wave-number is given by
$$k = k_0(\Lambda-1)^{\frac{1}{2}} (m_e/m_p)^{\frac{1}{2}},$$
where m_e/m_p is the ratio of the mass of the electron to that of the proton. Equation (38) becomes now
$$k^2\eta^2 = k_0^2[(m_e/m_p)(\Lambda-1)\eta^2 - (\partial\lambda_i/\partial\xi)\eta^3]. \tag{43}$$

If we follow through the calculation as before we find now that
$$\varpi_{\max\max} \doteqdot (8\,.\,2^{\frac{1}{2}}/3\pi\,.\,3^{\frac{1}{2}})\, k_0 \sigma^{(e)} \nu_1^{-1} [(m_e/m_p)\, \epsilon^{(p)}]^{\frac{3}{2}}, \tag{44}$$
where $\sigma^{(e)}$ is the velocity dispersion among the electrons and $\epsilon^{(p)}$ describes the anisotropy of the distribution function of the proton velocities.

Now ellipsoidal velocity distributions can occur in plasma flows where the density and temperature are uniform. If the density and/or the temperature vary from point to point then there is a further contribution present in the distribution function, of the form
$$\frac{\epsilon}{\sigma^4}(lu+mv+nw)\,\Xi\left(\frac{u^2+v^2+w^2}{\sigma^2}\right), \tag{45}$$

where the ϵ indicates that this contribution is of the same order as the ellipticity of the functions we have discussed so far. The new term is centrally anti-symmetrical; one readily verifies that at zero phase velocity it makes imaginary contributions of order ϵ to I_1, I_{vv}, I_{vw} and I_{ww}, and real contributions of the same order to I_v and I_w.

But in a plasma with an ellipsoidal velocity distribution the wave of largest wave-number has its wave-normal along the shortest axis of the velocity ellipsoid. Taking coordinates axes parallel to the principal axes we have

$$\mathscr{I}_{vv} = I_{vv} = \gamma^2/\alpha^2 \quad \text{and} \quad \mathscr{I}_{ww} = I_{ww} = \beta^2/\alpha^2,$$

while $I_{vw} = 0$ and $I_v = I_w = 0$. If corrections are needed to allow for the effect of the centrally anti-symmetrical contribution, then $\mathscr{I}_{vv}$ and $\mathscr{I}_{ww}$ become, respectively, $(\gamma^2/\alpha^2) + i\epsilon k_{vv}$ and $(\beta^2/\alpha^2) + i\epsilon k_{ww}$, correct to the first order in ϵ, where $i\epsilon k_{vv}$ and $i\epsilon k_{ww}$ are the contributions made to I_{vv} and I_{ww} by the anti-symmetrical part.

The new value for λ becomes

$$\lambda = (\gamma^2/\alpha^2) + i\epsilon k_{vv} \tag{46}$$

at $q = 0$. At the neighbouring real phase velocity $q = \xi$

$$\lambda = (\gamma^2/\alpha^2) + i\epsilon k_{vv} + \Lambda'\xi = (\gamma^2/\alpha^2) + i[\epsilon k_{vv} + \tfrac{1}{2}\nu_1 \pi \Delta^{\frac{1}{3}} \xi]. \tag{47}$$

In the presence of the anti-symmetrical component the location of the phase velocity where $\lambda = \gamma^2/\alpha^2$ is shifted to

$$q = \xi = -2\epsilon k_{vv}/\pi\nu_1 \Delta^{\frac{1}{3}} \tag{48}$$

on the real axis near $q = 0$. The calculation of the most unstable wave-number and of the corresponding amplification rate can be carried out as before. The only effect of the centrally anti-symmetrical part is therefore to give a small real part to the phase velocity of the most unstable mode.

4. Transverse instabilities in a fluid flow

We now consider a typical steady flow in an electron plasma in which the density $\rho(\mathbf{r})$, the systematic velocity $\mathbf{U}(\mathbf{r})$ and the r.m.s. thermal velocity $\sigma(\mathbf{r})$ all depend on the position $\mathbf{r}$. Let the frame of reference be so chosen that $|\mathbf{U}(\mathbf{r})| \ll \sigma(\mathbf{r})$ in the region of interest. Suppose first that individual electron velocities are randomized only by a process which allows them to persist for a period with expectation value τ. Let this process be such as to re-establish a locally isotropic velocity distribution. Then if $n(\mathbf{r}, \mathbf{u})\, d\mathbf{r}\, d\mathbf{u}$ is the *number* of electrons in the volume $d\mathbf{r}\,d\mathbf{u}$ of six-space, the Boltzmann equation becomes

$$(\mathbf{u}\,.\,\nabla)\, n = \frac{1}{\tau}\left[\frac{\rho}{m\sigma^3}\,\Phi\left\{\frac{|\mathbf{u}-\mathbf{U}|^2}{\sigma^2}\right\} - n\right]. \tag{49}$$

The solution of this equation is

$$n = \frac{\rho}{m\sigma^3}\,\Phi\left\{\left|\frac{\mathbf{u}-\mathbf{U}}{\sigma}\right|^2\right\} - \tau(\mathbf{u}\,.\,\nabla)\left[\frac{\rho}{m\sigma^3}\,\Phi\left\{\left|\frac{\mathbf{u}-\mathbf{U}}{\sigma}\right|^2\right\}\right], \tag{50}$$

correct to the first order in τ. To the same accuracy one can express the solution in the form

$$n = \frac{\rho}{m\sigma^3}\,\Phi\left\{\left|\frac{\mathbf{u}-\mathbf{U}\,(\mathbf{r}-\mathbf{u}\tau)}{\sigma}\right|^2\right\}$$

$$+\ \text{an anti-symmetrical contribution of order } \tau. \tag{51}$$

Only the centrally symmetrical part $n^{(s)}$ of n is of interest in a calculation of the instability rate, and we have

$$n^{(s)} = \frac{\rho}{m\sigma^3}\Phi\left\{\left|\frac{\mathbf{u}.(I+\tau\nabla\mathbf{U})-\mathbf{U}}{\sigma}\right|^2\right\}, \tag{52}$$

again correct to order τ, where I is the unit dyadic. Axes can be chosen so that at any desired point, say the origin, the tensor $I+\tau\nabla\mathbf{U}$ takes the form

$$I+\tau\nabla\mathbf{U} = \left|\begin{array}{ccc} 1+\tau\dfrac{\partial U}{\partial x} & \frac{1}{2}\tau\omega_3 & -\frac{1}{2}\tau\omega_2 \\ -\frac{1}{2}\tau\omega_3 & 1+\tau\dfrac{\partial V}{\partial y} & \frac{1}{2}\tau\omega_1 \\ \frac{1}{2}\tau\omega_2 & -\frac{1}{2}\tau\omega_1 & 1+\tau\dfrac{\partial W}{\partial z} \end{array}\right|. \tag{53}$$

Here U, V and W are the components of the velocity vector $\mathbf{U}$ and ω_1, ω_2 and ω_3 the components of the vorticity $\boldsymbol{\omega}$. With this choice of co-ordinate axes we can write (52) in the form

$$n^{(s)} = \frac{\rho}{m\sigma^3}\Phi\left[\frac{1}{\sigma^2}\left\{u^2\left(1+2\tau\frac{\partial U}{\partial x}\right)+v^2\left(1+2\tau\frac{\partial V}{\partial y}\right)+w^2\left(1+2\tau\frac{\partial W}{\partial z}\right)\right\}\right.$$
$$\left. + \text{linear terms in } u, v, w + \text{term independent of } u, v, w\right], \tag{54}$$

correct to the first order in τ. The quadratic terms in u, v and w fix the ellipsoidal nature of the velocity distribution; the remaining terms only serve to fix the local mean velocity. Purely rotational contributions to the tensor $I+\tau\nabla\mathbf{U}$ thus have no influence on the ellipticity of the velocity distribution. In fact the distribution is ellipsoidal if and only if $\partial U/\partial x$, $\partial V/\partial y$ and $\partial W/\partial z$ are not all equal. This is also the necessary and sufficient condition that the pressure tensor shall be anisotropic, so that viscous effects may arise in the fluid. In fact, whenever we expect viscous effects in a fluid we can also expect transverse instabilities in the corresponding plasma.

Let the x, y and z-axes be chosen so that $\partial U/\partial x \geqslant \partial V/\partial y \geqslant \partial W/\partial z$. It follows from (54) that the velocity ellipsoid then has principal axes in the ratio

$$1-\tau\,\partial U/\partial x : 1-\tau\,\partial V/\partial y : 1-\tau\,\partial W/\partial z \equiv \alpha:\beta:\gamma. \tag{55}$$

The largest unstable wave-number becomes

$$k_{\max} = k_0(\gamma^2/\alpha^2-1)^{\frac{1}{2}} = k_0[2\tau(\partial U/\partial x-\partial W/\partial z)]^{\frac{1}{2}} \tag{56}$$

and the maximum instability rate becomes

$$\varpi_{\max\max} = \frac{8.2^{\frac{1}{2}}}{3\pi.3^{\frac{1}{2}}}\frac{k_0\sigma\tau^{\frac{3}{2}}}{\nu_1}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{\frac{3}{2}}. \tag{57}$$

The instability may be expected to occur if the relaxation time τ much exceeds ϖ_{mm}^{-1}, the e-folding time of the most unstable wave. A further condition must be imposed because we are here trying to apply to a non-uniform medium some results originally derived for a uniform plasma. Physically this seems to be reasonable as long as the gradients of ρ, σ and $\mathbf{U}$ are small enough. A typical electron should not, during a period ϖ_{mm}^{-1}, be able to travel from a given point in the medium to another where ρ, σ or $\mathbf{U}$ is significantly different. This implies, for example, that one needs

$$\frac{\sigma}{\varpi_{mm}}\left\{\left|\frac{\partial U}{\partial x}\right|+\left|\frac{\partial V}{\partial y}\right|+\left|\frac{\partial W}{\partial z}\right|\right\} \ll \sigma, \tag{58}$$

or

$$\left|\frac{\partial U}{\partial x}\right|+\left|\frac{\partial V}{\partial y}\right|+\left|\frac{\partial W}{\partial z}\right| \ll \varpi_{mm}, \tag{59}$$

in order that the change in local mean velocity may be small compared with the typical thermal velocity.

Once excited, the instability will tend to reduce the time of free flight of the electrons below what one would estimate if one allowed only for the randomizing effect of collisions. In fact one surmises that the amplitude of the resultant disturbance becomes such that $\varpi_{mm}^{-1} \doteqdot \tau$, or

$$\tau^{\frac{5}{2}} \doteqdot \frac{3\pi \,.\, 3^{\frac{1}{2}}}{8 \,.\, 2^{\frac{1}{2}}}\frac{\nu_1}{k_0\sigma}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{-\frac{3}{2}} \doteqdot \frac{1{\cdot}5\nu_1}{k_0\sigma}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{-\frac{3}{2}},$$

and the effective time of free flight becomes

$$\tau \doteqdot \frac{1{\cdot}2\nu_1^{\frac{2}{5}}}{(k_0\sigma)^{\frac{2}{5}}}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{-\frac{3}{5}}. \tag{60}$$

One can now calculate the effective coefficient of viscosity μ corresponding to this τ. One possible definition (see, for example, Prandtl & Tietjens, 1934) is in terms of the pressure tensor P, by the equation

$$-\mu[\nabla\mathbf{U}+(\nabla\mathbf{U})'-\tfrac{2}{3}I\operatorname{div}\mathbf{U}] = P-\bar{p}I, \tag{61}$$

where $(\nabla\mathbf{U})'$ is the transpose of $\nabla\mathbf{U}$, I is as before the unit dyadic, and

$$\bar{p} = \tfrac{1}{3}(p_{xx}+p_{yy}+p_{zz}),$$

the mean pressure. In the present case it follows from (54) that

$$p_{xx} = \rho\sigma^2(1-2\tau\,\partial U/\partial x)$$

and so on for p_{yy}, p_{zz}, and that

$$\bar{p} = \rho\sigma^2(1-\tfrac{2}{3}\tau\operatorname{div}\mathbf{U}). \tag{62}$$

On equating xx-components of (61) and using (60) one finds that

$$\mu = \rho\sigma^2\tau = \frac{1{\cdot}2\rho\sigma^2\nu_1^{\frac{2}{5}}}{(k_0\sigma)^{\frac{2}{5}}}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{-\frac{3}{5}}. \tag{63}$$

On substituting $k_0^2 = 4\pi Ne^2/mc^2$, and putting $\nu_1 = (2/\pi)^{\frac{1}{2}}$ for a Gaussian distribution, one then derives from (63) that

$$\mu = 1{\cdot}7\,\frac{m^{\frac{1}{5}}\,\bar{p}^{\frac{4}{5}}}{r_e^{\frac{1}{5}}}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{-\frac{3}{5}} = \frac{1{\cdot}7(kT)^{\frac{4}{5}}}{m_e^{\frac{3}{5}}r_e^{\frac{1}{5}}}\,\rho^{\frac{4}{5}}\left(\frac{\partial U}{\partial x}-\frac{\partial W}{\partial z}\right)^{-\frac{3}{5}}, \tag{64}$$

where $e^2/mc^2 = r_e$ is the classical radius of the electron. This form of μ is obviously different in nature from that for an ordinary gas. Two notable points are that μ

is now proportional to $\rho^{\frac{4}{5}}$, and that it is dependent on the local state of motion of the medium. Clearly other transport coefficients, such as the thermal and electrical conductivities, will be changed in a similar way.

5. Discussion

The calculation of the previous section raises several problems. Thus the transverse instabilities invoked can be shown to occur in a plasma containing no external fields, but there may be situations where such fields are necessarily introduced. Consider, for example, the determination of the structure of a collision-free shock in which the entropy gain is due to these instabilities. Unstable waves can arise among the electrons or the protons. But the proton instability takes much longer to develop than that of the electrons, and so, as in a number of other plasma phenomena, the energy dissipation occurs essentially among the electrons, which are slowed down first. The protons are then dragged back by electrostatic forces. Thus, in finding the structure of the shock, one must remember that the electron instability needs to be maintained in the presence of an electrostatic field. In the downstream part of the structure there will also be magnetic fields present, produced by the unstable waves further upstream, and swept down from there. To know how long these fields persist one should find out what happens to the unstable waves at finite amplitude.

Lastly we expect that the final effect of a transverse instability is to make the velocity distribution of the electrons locally isotropic. To check this one needs, once again, to study finite amplitude effects. But even without such a study it is clear that there are some simple cases in which the required re-arrangement of velocities is not straightforward. For example, let the local mean velocity be given by

$$U = \tfrac{1}{2}\zeta(x+z), \quad V = 0, \quad W = -\tfrac{1}{2}\zeta(x+z)$$

where $\zeta > 0$. This describes a solenoidal shear flow parallel to the (x, z)-plane. The resultant velocity distribution is locally ellipsoidal, and has its shortest axis parallel to Ox and its longest axis parallel to Oz. The most unstable plasma wave then has its wave-normal parallel to Ox, its vector potential parallel to Oz and its magnetic field parallel to Oy. The forces on the charged particles due to such a wave all necessarily act parallel to the (x, z)-plane. No complete redistribution of velocities can occur on account of this wave alone. In fact after the velocities have been redistributed in the (x, z)-plane a further wave must become unstable, with a wave vector normal to that plane, and must complete the redistribution.

In both the collision free shock and the shear flow it seems that the physical details describing the plasma waves are likely to be rather complicated.

REFERENCES

KAHN, F. D. 1962 *J. Fluid Mech.* **14**, 321.

PRANDTL, L. & TIETJENS, O. G. 1934 *Hydro and Aeromechanics*, p. 258. New York and London: McGraw-Hill.